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Flow of geometries and instantons on the null orbifold

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ABSTRACT: We study condensation of twisted sector states in the null orbifold geometry. As the singularity is time-dependent, we probe it using D-Instantons. We present evidence that the null-orbifold flows to the $\mathbb{Z}_N$ orbifold. We also comment on the subtleties of quantizing the closed superstring in this background.

KEYWORDS: Tachyon Condensation, Black Holes in String Theory.

Contents

1. Introduction

The treatment of time-dependent backgrounds and spacelike singularities remains among the main puzzles of string theory. The importance of the problem has led to a considerable amount of work, and some progress has been made using perturbative and non-perturbative techniques, such as exact CFT's, AdS/CFT etc. ([1 − 36]). Examples of the goals of this program include understanding the quantum state at the big-bang singularity, or at the singularity of the black hole [37].

In particular, the role of α' corrections vs. perturbative g_s corrections vs. non-perturbative stringy corrections, at such singularities, is unclear. It is likely that all are needed in order to understand the singularity in detail. This is the case in more familiar stringy singularities, such as the conifold, for which key effects are understood by quantitative non-perturbative effects, such as wrapped D-branes becoming light [38], but a more detailed understanding uses LST or double-scaled LST [39]. The latter focuses on a smaller set of degrees of freedom localized near the singularity (and is a solvable CFT with a varying g_s), which clarifies where and how non-perturbative effects set in, as well as being computationally useful.

Other effects which are likely to be important are effects beyond 2nd quantized string theory. Already at the level of the Einstein-Hilbert action generic singularities exhibit a strong mixing property — the BKL dynamics (for a review of its recent appearance in supergravity see [34]). There are no proposals for a tractable stringy formalism to deal with such mixing (beyond the low energy effective action, which breaks down close to the singularity).

We list below some of the motivation for exploring α' effects around the singularity:

1. In [31], a search was carried out for remnants of the black hole (BH) singularity at large N but weak 't-Hooft coupling. No indications of the singularity were found. This might suggest that the singularity is resolved by α' corrections.

2. α' effects already exhibit interesting and unexpected behavior near the singularity. In [33] it was shown that already at sphere level a non-commutative geometry like structure appears. The latter delocalized the twisted sector states over large distances in spacetime. In particular, in [29] it was shown that twisted sector pair creation occurs near the singularity.

3. For the non-rotating extremal BTZ black hole in AdS_3, one can construct candidates for the microstates of the BH (for a recent review see [40]). The proposed microstates of the BH are solutions of Einstein-Hilbert action, and in particular do not require higher g_s corrections. However, they do require α' corrections in order, for example, to understand the $\mathbb{Z}_N$ singularities which occur in some of these configurations.

4. It was recently shown that the singularity and horizon structure of a class of supersymmetric black holes changes significantly by tree level or 1-loop higher order curvature corrections to the effective action [41, 42].

5. Although not directly related yet, a problem of singularities in causal structure also appears in purely 2D field theory context. The Coulomb branch of SU(2) (4,4) gauge theory in two dimensions [43] with a single quark flavor has a metric which is not positive definite in the IR.[1] Since the model is the IR of a perfectly well defined unitary supersymmetric field theory, this problem has to be resolved within the 2D field theory. One usually says that the correct degrees of freedom were not identified properly in the IR (already in terms of the 2D field theory).

[1]The metric is believed to receive only a 1-loop corrections in perturbation theory.

We are interested in mapping what string theory considers to be small deformations of a singularity, in our case the null orbifold singularity [1, 7, 8, 13, 44], as a step towards understanding its large deformation, which might be relevant for its resolution. For the null orbifold, we will explore the relation between it and familiar $\mathbb{Z}_N$ orbifolds that possess a mild (and well understood in string theory) time like singularity.

One can also consider a more detailed role that "near by geometries" can play. Consider for example the relation between the fuzzball states of [40] and the BTZ geometry. One way to reconcile the validity of the two descriptions is examine the amount of mixing the micro-states undergo under any attempt to probe them. Since they differ from each other only over small length scales,[2] they clearly mix under any such perturbation. This implies that the effective geometry may not be that of the microstates but something else — perhaps for some purposes it is the original BTZ black hole geometry. The situation might be analogous to that of some field theories which at zero temperature exhibit spontaneous symmetry breaking, but exhibit symmetry restoration at finite temperature (above a threshold). The microstates are analogous, in this very rough analogy, to the true vacua, and the finite temperature minimum in the origin is analogous to the original black hole singularity, which dominates the dynamics once mixing is taken into account. In both cases a complicated enough process, with enough energy, will be dominated by the symmetric phase (=the background with the black hole singularity) although most pure states (and in particular the ground state) are in the broken symmetry phase (which is like the micro-state description). To go from the symmetric phase to the broken phase one usually condenses a tachyon (at zero temperature), and hence we would like to explore the analogues of these tachyons in the case of the spacelike (or null) singularity.

The null-orbifold singularity has a very concrete relation to the $\mathbb{Z}_N$ singularity. Already in [13] this relation was touched upon, and we develop it further here. In section 2 we show how precisely the two-cone null-orbifold is a large N limit of the better understood single cone $\mathbb{C}/\mathbb{Z}_N$ orbifold (as well as the subtleties associated with this limit). In section 3 we discuss the action of $D(-1)$ branes in this background. In section 4 we explore the transition from the null orbifold towards the $\mathbb{Z}_N$ orbifolds, after condensing an N-twisted sector state in the null-orbifold.

This situation can be summarized in the following diagram

$$\begin{tikzcd} \mathbb{R}^1 \times \mathbb{C}/\mathbb{Z}_n \arrow[r, "n\to\infty", "\infty\text{ boosted}"'] \arrow[d, "\text{tachyon}"', "\text{sector } k"] & \mathbb{R}^{1,2}/\text{Null} \arrow[d, "\text{tachyon}"', "\text{sector } m"] \\ \mathbb{R}^1 \times \mathbb{C}/\mathbb{Z}_{\frac{n}{k}=m} \arrow[r, "{n,k\to\infty,\ \frac{n}{k}=m}", "\infty\text{ boosted}"'] & \mathbb{R}^1 \times \mathbb{C}/\mathbb{Z}_m \end{tikzcd} \tag{1.1}$$

The left downward point arrow is the flow of [47]. The upper rightward pointing arrow is section 2. The right downward pointing arrow is section 4 and our main conclusion of the paper, which presents evidence that upon condensation of an N twisted sector mode of the null-orbifold, a $\mathbb{Z}_N$ singularity appears.[3]

[2] We would like to thank S. Ross for a discussion of this point.

[3] Although in the extreme boost limit, as we will discuss later.

Section 5 contains a summary and conclusions.

Further details on the $\mathbb{Z}_N \to$ *null orbifold* are provided in appendix A. Appendix B quantizes the RNS string on the null orbifold and shows the emergence of a logarithmic CFT.

As we completed this, two papers appeared which discuss related models from a different perspective [50 – 52].

2. The null-orbifold and $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold

String theory in orbifolds of the form $(\mathbb{R}^{1,2}/\Gamma) \times \mathcal{C}^{\perp}$ with Γ generating a group isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_N$ were extensively studied. The $\mathbb{Z}_N$ orbifolds, (where Γ is in the elliptic class of $\mathrm{SO}(1,2)$) are well understood [45 – 48] (for a review see [49]), and the geometry is a cone perpendicular to the time direction. The null-orbifold studied in [7, 13, 44] is generated by Γ in the parabolic class of $\mathrm{SO}(1,2)$, and the geometry consists of two three dimensional cones with a common tip and a singular plane crossing the tip. Unlike the $\mathbb{Z}_N$ orbifolds the null-orbifold is a singular time-dependent background.[4] The singularity at the origin of the cones is still not completely understood.

In this section we review the quantization of the $\mathbb{Z}_N$ and the null orbifold. We introduce the construction of the latter from the former by a infinite boost. The result we obtain is that the $\mathbb{Z}_N$ orbifold converges to the two-cone null orbifold. This is shown using both the classical geometry and the 1st quantized string.

2.1 Classical results (geometry)

To describe orbifolds of $\mathbb{R}^{1,2}$ we take the coordinates x^0, x^1, x^2 on $\mathbb{R}^{1,2}$, with the flat metric $ds^2 = -d(x^0)^2 + d(x^1)^2 + d(x^2)^2$ and consider the Killing vector:

$$J(a,b) = bJ^{02} + aJ^{12} \tag{2.1}$$

where $J^{02} = x^0\partial_2 + x^2\partial_0$ is a boost and $J^{12} = x^1\partial_2 - x^2\partial_1$ is a rotation.

For $b < a$, $J(a,b)$ is in the elliptic class, and is conjugate to $J(\sqrt{a^2-b^2}, 0)$ using some Lorentz transformation M. The null orbifold is given by $a = b$. Choosing $\sqrt{a^2-b^2} = 1/N$, the generator of the null orbifold, $a = b$, is identified as the limit $N \to \infty$ of a sequence $M_N J(1/N, 0) M_N^{-1}$, where M_N is an N-dependent Lorentz transformation. This Lorentz transformation is singular when $N \to \infty$. It is useful to write the explicit form of M_N.

$$M_N = \begin{pmatrix} \frac{a}{\sqrt{a^2-b^2}} & 0 & \frac{b}{\sqrt{a^2-b^2}} \\ \frac{b}{\sqrt{a^2-b^2}} & 0 & \frac{a}{\sqrt{a^2-b^2}} \\ 0 & -1 & 0 \end{pmatrix}$$

$$J(a,b) = M_N J\left(\frac{1}{N}, 0\right) M_N^{-1}\,. \tag{2.2}$$

For the pure rotation case ($b = 0$) we will use the familiar convention:

$$\left(x^0\,,\ z\,,\ \bar{z}\right) \cong \left(x^0\,,\ e^{\frac{2\pi i}{N}} z\,,\ e^{-\frac{2\pi i}{N}} \bar{z}\right) \qquad \text{with } z = \frac{x^1 + ix^2}{\sqrt{2}}\,. \tag{2.3}$$

[4] In spite of the fact it possesses a null isometry.

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The generator of the null Orbifold twisting operator acts on the light cone coordinates

$$x^+ = \frac{x^0 - x^1}{\sqrt{2}} \qquad x = x^2 \qquad x^- = \frac{x^0 + x^1}{\sqrt{2}} \tag{2.4}$$

by

$$J^{\text{null}} = a \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} . \tag{2.5}$$

We shall follow the convention of [13] in defining the null orbifold by choosing $a = \frac{\nu}{2\pi\sqrt{2}}$ and the identification takes the following form

$$\begin{aligned} x^+ &\sim x^+ \\ x &\sim x + \nu x^+ \\ x^- &\sim x^- + \nu x + \frac{\nu^2}{2} x^+ . \end{aligned} \tag{2.6}$$

A fundamental domain of the space looks like two 3 dimensional cones emanating from $x^+ = x = 0$, one towards $x^+ > 0$ and one towards $x^+ < 0$, and two codimension 1 cones in the plane $x^+ = 0$ pinching at $x = x^- = 0$. Any two null orbifolds (differing by the value of ν) are related by boosts in the x^1 direction.

We show how by the procedure described above, of boosting by M_N, we may actually understand geometrically that there is a singular limit which relates the $\mathbb{Z}_N$ space with the null orbifold space. This isn't straightforward as the $\mathbb{Z}_N$ is a one cone space and the null orbifold has two cones (that are not contained in the singular plane). Indeed for any finite N, we remain with an one cone fundamental domain. Some of the orbits in the x^+, x, x^- coordinates for large N are plotted in figure 1. One confirms the impression from the figure, that the orbits become localized around a fixed x^+ with a spread in x^+ which is $1/N$ that of x^-. At $N \to \infty$ we observe that the slopes go to infinity. In this limit the orbits are contained in the x x^- plane, at fixed x^+. The infinity slopes orbits are exactly the parabolas of the null orbifold. Hence the single cone orbifold maps onto the two cone null-orbifold geometry.

2.2 Hilbert space: untwisted sector

As is usual in orbifolds, the untwisted wave functions are projections of the wave functions of the covering space. The latter are plane waves on $\mathbb{R}^{1,2}$, and the untwisted sector of the orbifold (focusing on scalar functions) is [13]

$$\Psi^{orb.}_{k,s} = \int_{-\infty}^{\infty} ds\, e^{2\pi s\left(il + \hat{J}\right)} \psi_k(x) , \qquad l \in \mathbb{Z} \tag{2.7}$$

where $\hat{J}$ is the action of the null boost generator on the function ψ_k (the wave function in flat space).

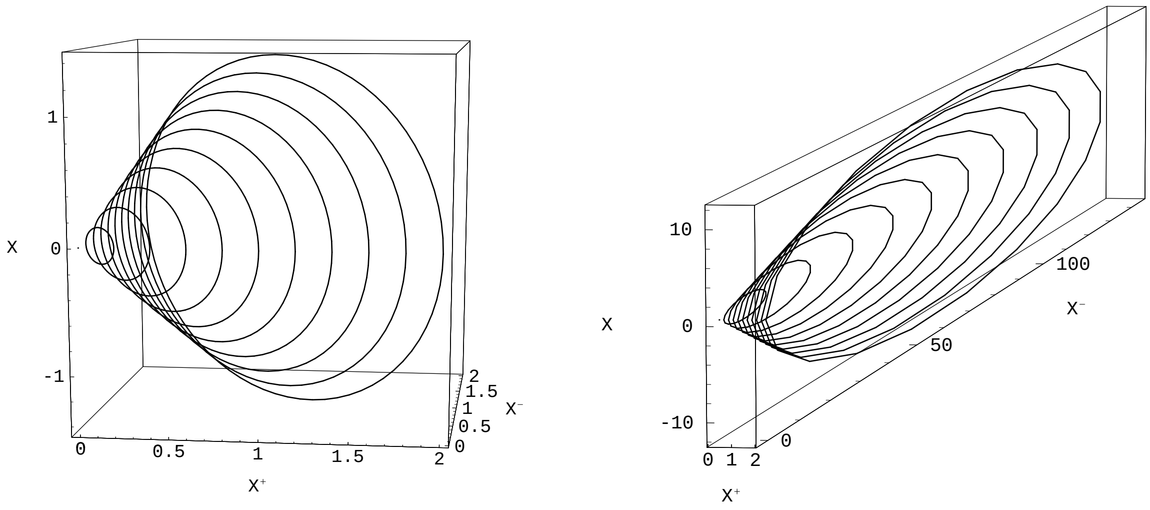

Figure 1: Particular orbits of the boosted cone for N=3 (left) and N=6 (right). The orbits are generated from points on the x^+ axis in the segment $x^+ \in (0, 2)$ by a continuous action of the orbifold generator $J(a, b)$. For large N the orbits are stretched in the $x - x^-$ plane, such that at the limit $N \to \infty$ they are confined to that plane (i.e x^+ is constant within a single orbit). For the limit $N \to \infty$ we expect the orbits to close only at infinity, thus dividing the single cone to a double cone.

The formula for the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold is similar. The invariant wave functions of the elliptic orbifold are

$$\Psi^N_{k,l} = \frac{N}{2\pi} e^{-ik^0x^0} \oint_0^{2\pi} d\theta e^{ik\bar{z}e^{i\theta} + i\bar{k}ze^{-i\theta} + iN\theta l}\,, \qquad l \in \mathbb{Z}$$
$$\text{with}\,,\ k = \frac{k^1 + ik^2}{\sqrt{2}} \qquad z = \frac{x^1 + ix^2}{\sqrt{2}}\,. \tag{2.8}$$

These integrals can be evaluated in terms of Bessel functions:

$$\Psi^N_{k,l} = Ne^{-ik^0x^0 + iN\phi l} J_{Nl}(2u)\,, \qquad ue^{i\phi} \equiv -i\bar{k}z\,. \tag{2.9}$$

It is easy to verify the completeness of this basis. We also choose k to be real.

For the null-orbifold the integration in (2.7) is gaussian (The operator $\hat{J}^{\text{null}}$ is nilpotent of order 3), and the wave-function matches the results of [13] (after fixing the phase and taking $\nu = 2\pi$ until the end of the section):

$$\Psi^{\text{null}}_{k,l} = \frac{\exp\left[i\left(-k^+x^- - k^-x^+ + \frac{(l-k^+x^2)^2}{2k^+x^+}\right)\right]}{\sqrt{2\pi}\sqrt{ik^+x^+}}\,, \qquad l \in \mathbb{Z}$$
$$\text{where } k^\pm = \frac{1}{\sqrt{2}}(k^0 \mp k^1) \qquad x^\pm = \frac{1}{\sqrt{2}}(x^0 \mp x^1)\,. \tag{2.10}$$

This is the wave function on the three dimensional cones. On the singular co-dimension 1 cones, it is a distribution.

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We have shown that one can take the limit of the geometry. However, since we are interested in a CFT statement, it is more meaningful to show that the limit of the set of wave functions on the single cone R^2/Z_N is the set of all wave functions on the null orbifold. Under an M_N boost, the wave-function transforms as:

$$\begin{aligned}\Psi^{(a,b)}_{k,l}(x) &\equiv \int_{-\infty}^{\infty} ds\, e^{2\pi\left(il+\hat{J}(a,b)\right)}\psi_k(x)\\ &\propto \int_{-\infty}^{\infty} ds\, \hat{M}_N e^{2\pi s\left(il+\hat{J}(\frac{1}{N},0)\right)}\hat{M_N^{-1}}\psi_k(x) \propto \Psi^{N}_{\widetilde{k},l}(M_N^{-1}x)\end{aligned} \tag{2.11}$$

where $\widetilde{k}_\mu = k_\nu (M_N)^\nu{}_\mu$. In the limit $N \to \infty$, after normalizing the wave functions, we expect to find that:

$$\Psi^{\rm Null}_{k,l}(x) = \lim_{N\to\infty} \Psi^N_{kM_N,l}(M_N^{-1}x)\,. \tag{2.12}$$

To demonstrate that this limit is well defined (and not just formal) we will show it explicitly using the wave-function (2.9) and (2.10). The limit is defined so that the parameters a, b in M_N approach their final (common) value symmetrically[5] keeping $a > b$

$$a = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}N^{-2} + O(N^{-4}) \qquad b = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}N^{-2} + O(N^{-4})\,. \tag{2.13}$$

Applying this transformation to the wave-functions (2.9) in the large N limit we find (further details are provided in appendix A):

$$\begin{aligned}\Psi^{(a,b)}_{k;l}(x) = \frac{(\text{sign}(k^+x^+))^{Nl}}{\sqrt{2\pi i x^+k^+}}\Bigg\{&\exp\left[i\frac{\left(l-x^2k^+ + x^+k^2\right)^2}{2x^+k^+} + ik^2x^2 - ik^+x^- -\right.\\ &\left.- ik^-x^+ + O(N^{-1})\right] +\\ &+ (-)^{Nl+1}\exp\left[-i\frac{(l+x^2k^+ - x^+k^2)^2}{2x^+k^+} - ik^2x^2 -\right.\\ &\left.- i2N^2k^+x^+ + O(N^{-1})\right]\Bigg\}\end{aligned} \tag{2.14}$$

k^2 may be set to zero using a Lorentz transformation. Taking the limit $N \to \infty$ the second, term is zero[6] and the first term reduces exactly to the null-orbifold wave function[7] (2.10). Thus completing our proof.

2.3 Hilbert space: twisted sectors

The authors of [13] showed that the zero mode wave functions in the twisted sectors of the null-orbifold are (in our conventions of normalizing the wave functions):

$$\Psi^w_{m,p^+,J} = \sqrt{\frac{1}{2\pi i p^+x^+}}\exp\left[-ip^+x^- - i\frac{m^2}{2p^+}x^+ + i\frac{p^+}{2x^+}\left(x+\frac{J}{p^+}\right)^2 - i\frac{w^2(x^+)^3}{6(\alpha')^2p^+}\right] \tag{2.15}$$

[5]This is a necessary requirement. For other prescriptions we have not been able to show that one obtains the null orbifold wave functions.

[6]To see that, it should be considered as a distribution: any integral with a well behaved function vanishes in the limit. The exact statement is explained in appendix A.

[7]The sign in front is irrelevant as Nl can be chosen even throughout.

with $J \in \mathbb{Z}$ and m is the three dimensional mass given by the on shell condition:

$$2p^+p^- = m^2 = -\frac{4}{\alpha'} + \vec{p}_\perp^{\,2}\,.$$

One observes that this wave function is the invariant combination of wave functions describing a particle in the presence of an extremal configuration of electric-magnetic fields $E = \pm B$ (in analogy to [26]). The absolute value of this field is proportional to the twist parameter.

We now write the twisted sector wave functions of the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold in the same manner by considering a particle in the presence of a magnetic field.[8] The most convenient covering space wave functions are radial strips:

$$\Psi^{k/N}_{p^0,j,n_r} = e^{ij\phi} e^{-ip^0x^0} R_{|j|,n_r}\left(\frac{k}{N\alpha'}r^2\right) \tag{2.16}$$

where r, ϕ are the usual polar coordinates on the plane, $j \in \mathbb{Z}$ and the radial function is given by

$$R_{|j|,n_r}(\xi) = e^{-\xi/2}\xi^{|j|/2}F\big(-n_r, |j|+1, \xi\big) \tag{2.17}$$

F is the degenerate (confluent) hypergeometric function. The orbifold invariance condition is simple in these coordinates and is given by $j \in N\mathbb{Z}$. The wave function describes a particle with mean distance of $\sim n_r\sqrt{\alpha'}$ from the origin and the wave function has $\sim j$ oscillations. The conformal weight (energy) of this state is

$$E = -\frac{\alpha'}{4}(p^0)^2 + \frac{k}{N}\left(n_r + \frac{1}{2}(|j| - j + 1)\right). \tag{2.18}$$

The relation to the usual states of the CFT, which are given by acting with creation quasi zero modes, is the following:

$$L_0 = E \qquad \frac{k}{N}j = L_0 - \tilde{L}_0\,. \tag{2.19}$$

Using these relations one may obtain the CFT meaning of n_r

$$\frac{k}{N}n_r = \frac{\alpha'}{4}(p_0)^2 + L_0 - \frac{1}{2}\left(\left|L_0 - \tilde{L}_0\right| - (L_0 - \tilde{L}_0) + 1\right). \tag{2.20}$$

In order to boost the wave function, we boost the coordinates as in (2.12) but the quantum numbers are more subtle. In order to match the quantum numbers of the boosted wave function with the quantum numbers of the null orbifold it is very suggestive to use the geometric interpretation of n_r. Indeed, we expect it to transform as $\sim (p^1)^2 + (p^2)^2$. The exact way is inferred from (2.18) and we simply replace n_r everywhere by

$$n_r \equiv \frac{\alpha' N}{4k}((p^1)^2 + (p^2)^2). \tag{2.21}$$

[8]For completeness, we derive this idea on a general orbifold of $\mathbb{R}^{1,2}$ in the next subsection.

The quantum number j/N (which is integer) will be mapped exactly to l as can already be seen from the fact the azimuthal part of the wave function (2.16) coincides with the untwisted wave function azimuthal part (2.9).

To recapitulate, the wave function of the twisted $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ which is ready to be boosted with appropriately chosen quantum numbers is

$$\Psi^{k,N}_{p^0,|p|,l} = e^{iNl\phi} e^{-ip^0 x^0} R_{|Nl|,\frac{\alpha' N}{4k}|p|^2} \left(\frac{k}{N\alpha'} r^2 \right). \tag{2.22}$$

In doing the boost it is convenient to move to Whittaker functions which are related to R according to

$$M_{\frac{|j|}{2}+n_r+\frac{1}{2},\frac{|j|}{2}}(z) = z^{1/2} R_{|j|,n_r}(z). \tag{2.23}$$

Using the M_N above, and the asymptotics of the Whittaker function one can show that (2.22) converges to (2.15).

2.4 First quantization of the bosonic string

Applying the quantization scheme introduced in [26] we quantize the bosonic string on $(\mathbb{R}^{1,2}/\Gamma) \times \mathbb{R}^{d-3}_{\perp}$. A more detailed discussion including quantization of the superstring is postponed to appendix B.

The worldsheet action and monodromies are:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, \eta_{\mu\nu} \left(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu \right) \\ X(\sigma + 2\pi, \tau) &= e^{2\pi w \mathcal{J}} X(\sigma, \tau) \end{aligned} \tag{2.24}$$

where $w \in \mathbb{Z}$ is the twisted sector number and $\mathcal{J}$ is matrix defined from the differential realization of (2.1):

$$\hat{J}(a,b) = X^\mu \mathcal{J}_\mu{}^\nu \frac{\partial}{\partial X^\nu}$$

The mode expansion in a twisted sector $w \neq 0$:

$$\begin{aligned} X^\mu(\sigma,\tau) \,=\, \left[e^{w\mathcal{J}\sigma}\right]_\nu{}^\mu X_z{}^\nu(x,p\,;\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left[\frac{e^{-i(n+iw\mathcal{J})(\sigma+\tau)}}{n + iw\mathcal{J}} \right]_\nu^{\ \mu} \alpha_n^\nu \,+ \\ +i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left[\frac{e^{i(n-iw\mathcal{J})(\sigma-\tau)}}{n - iw\mathcal{J}} \right]_\nu^{\ \mu} \tilde{\alpha}_n^\nu\,. \end{aligned} \tag{2.25}$$

With the zero-mode

$$\begin{aligned} \text{Null}: \quad X_z{}^\mu(x,p\,;\tau) \,&=\, \cosh\big(w\tau\mathcal{J}\big)^\mu{}_\nu x^\mu + \left[\mathbb{1} + \frac{1}{2}\cosh\big(w\tau\mathcal{J}\big)\right]_\nu^{\ \mu} \frac{2\alpha'\tau}{3} p^\nu \\ \text{Other}: \quad X_z{}^\mu(x,p\,;\tau) \,&=\, \cosh\big(w\tau\mathcal{J}\big)^\mu{}_\nu x^\mu + \left[(w\mathcal{J})^{-1}\sinh(w\mathcal{J}\tau)\right]_\nu{}^\mu \alpha' p^\nu\,. \end{aligned} \tag{2.26}$$

Where x^μ and p^μ satisfy the canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$.

Introduce the operators:

$$\alpha_0 = \sqrt{\frac{1}{2\alpha'}}(w\mathcal{J}x + \alpha' p) \qquad \tilde{\alpha}_0 = \sqrt{\frac{1}{2\alpha'}}(w\mathcal{J}x - \alpha' p)\,. \tag{2.27}$$

Although assigned subscript 0, these aren't real zero modes in general and may possess nonzero conformal weight as we discuss below. Hence, we will properly name them quasi zero modes. The commutation relations between the modes:

$$[\alpha_n, \alpha_m] = \delta_{n+m}(n + iw\mathcal{J})\,\eta\,, \qquad [\tilde{\alpha}_n, \tilde{\alpha}_m] = \delta_{n+m}(n - iw\mathcal{J})\,\eta\,, \qquad [\alpha_n, \tilde{\alpha}_m] = 0\,. \quad (2.28)$$

The Virasoro generators are

$$L_n = -\frac{1}{\alpha'}\oint \frac{dz}{2\pi i} z^{n+1} :\!\partial X \partial X(z)\!: = \frac{1}{2}\sum_m \,{}^\circ_\circ\, \alpha_m \eta \alpha_{n-m} \,{}^\circ_\circ + \delta_{n,0}a^X(w)\,. \quad (2.29)$$

For any choice of normal ordering scheme of the quasi zero modes the zero point energy in the bosonic sector is

$$a^X(w) = \frac{w^2}{4}\,\mathrm{Tr}(\mathcal{J}^2) + \left\langle \mathrm{vac}\left|\frac{1}{2}\eta_{\mu\rho}\alpha_0^\mu \alpha_0^\rho\right|\mathrm{vac}\right\rangle_w\,. \quad (2.30)$$

The correct choice of normal-ordering depends on the $\mathrm{SO}(1,2)$ class of the orbifold identification generator:

- Elliptic class: the quasi zero modes have positive and negative conformal weights. Therefore, we naturally choose the positive conformal weight to annihilate the twisted vacuum state. The collection of states generated by acting with the creation operators is the Hilbert space of a (1st quantized) particle in a uniform magnetic field.

- Hyperbolic class: the quasi zero modes have pure imaginary conformal weight, as discussed in [26] we should choose the reality conditions to be consistent with the commutation relations. This way, L_0 and $\tilde{L}_0$ are manifestly hermitian. The result is the Hilbert space of a particle in uniform electric field.

- Parabolic class: the quasi zero modes have conformal weight zero. The Hilbert space is of a particle in electric-magnetic fields which are equal in magnitude. One choice of quantization is that of (2.15).

3. $D(-1)$ instanton probes

3.1 The world-volume theory

We wish to examine the null orbifold deformed by some tachyon condensate. We shall probe the space with instantons ($D(-1)$ branes), following the discussion of [47, 53]. In this section we develop the technology needed, beginning with the non deformed null orbifold. The open string theory on the instantons is a matrix theory of the collective coordinates (fermionic and bosonic). We focus on the bosonic degrees of freedom, they are parameterized by the covering space coordinates:

$$(X^+, X^-, X) \in \mathbb{R}^{2,1}\,, \qquad (Y^3, Y^4, \dots Y^9) \in \mathbb{R}^7_\perp\,.$$

Under the projection each $D(-1)$ instanton has infinitely many images, we use Chan-Paton indices that span the adjoint of $\mathrm{U}(\infty)$ in the covering space. The null-orbifold projection should break the $\mathrm{U}(\infty)$ to $\mathrm{U}(1)^{\infty}$, even at the singularity. We use the orbifold projection:

$$\begin{aligned} Y^a_{i,j} &= Y^a_{i-1,j-1}\,, \qquad a = 3\dots 9 \\ X^+_{i,j} &= X^+_{i-1,j-1} \\ X_{i,j} &= X_{i-1,j-1} + \nu X^+_{i-1,j-1} \\ X^-_{i,j} &= X^-_{i-1,j-1} + \nu X_{i-1,j-1} + \frac{1}{2}\nu^2 X^+_{i-1,j-1}\,. \end{aligned} \tag{3.1}$$

These recursive equations are linear and can be easily solved. The solution can be neatly written using the following matrices (which also define an infinite closed algebra):

$$\begin{aligned} (\beta^m_l)_{i,j} &\equiv \frac{(i+j)^l}{2^l}\delta_{i,j-m} \\ [\beta^m_l, \beta^{m'}_{l'}] &= 2\sum_{p=0}^{l}\sum_{p'=0}^{l'}\left(\frac{m}{2}\right)^{l'-p'}\left(-\frac{m'}{2}\right)^{l-p}\beta^{m+m'}_{p+p'}\,\delta_{p'-p\in 2\mathbb{Z}+l'-l+1}\binom{l}{p}\binom{l'}{p'}\,. \end{aligned} \tag{3.2}$$

The collective coordinates (bosonic) fields which solve (3.1):

$$\begin{aligned} Y^a &= \sum_{m\in\mathbb{Z}} y^a_m \beta^m_0 \qquad X = \sum_{m\in\mathbb{Z}} x_m\beta^m_0 + \nu x^+_m \beta^m_1 \\ X^+ &= \sum_{m\in\mathbb{Z}} x^+_m \beta^m_0 \qquad X^- = \sum_{m\in\mathbb{Z}} x^-_m\beta^m_0 + \nu x_m\beta^m_1 + \frac{\nu^2}{2}x^+_m\beta^m_2 \end{aligned} \tag{3.3}$$

along with the reality conditions

$$(y^a_m)^* = y^a_{-m} \qquad (x^+_m)^* = x^+_{-m} \qquad (x^-_m)^* = x^-_{-m} \qquad (x_m)^* = x_{-m}\,.$$

The world volume low energy effective lagrangian of the instantons may be obtained by dimensionally reducing the 10d Super Yang Mills. The action on the world-volume is:[9]

$$\begin{aligned} \mathrm{S} &= -\frac{1}{2Z_0}\sum_{\mu,\nu=0}^{9}\mathrm{Tr}\left([X^\mu, X^\nu][X_\mu, X_\nu]\right) \\ &= \sum_{m+n+m'+n'=0}\Big[-y^a_m y^a_{m'} x^+_n x^+_{n'} mm' + x_n x^+_m x_{n'} x^+_{m'}(mm' - 2nm') + \\ &\qquad + 2x^+_m x^-_n x^+_{m'} x^+_{n'} nm' + \nu^2 x^+_m x^+_n x^+_{m'} x^+_{n'}\frac{mm'n^2}{4}\Big]\,. \end{aligned} \tag{3.4}$$

It is useful to represent the above action using real bosonic fields living on S^1, identifying the modes as Fourier coefficients for a real field on S^1 as in [53]:

$$u_m = \int_0^{2\pi}\frac{d\sigma}{\sqrt{2\pi}}\,\mathrm{U}(\sigma)e^{-im\sigma}\,, \qquad U = \left(X^+, X^-, X, Y^3, \dots Y^9\right) \tag{3.5}$$

[9] The normalization factor Z_0 (which is simply $\sum_{-\infty}^{\infty} 1$) is set to remove an overall infinite factor coming from the traces.

with the action:

$$S = \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[(X^+ \dot{Y^a})^2 - (X\dot{X^+})^2 + 2\dot{X}\dot{X}^+ X X^+ - 2(X^+)^2 \dot{X}^+ \dot{X}^- - \frac{\nu^2}{12}(\dot{X}^+)^4 \right]. \quad (3.6)$$

One may think of this action as the (analog of) T dual action, for D0 branes wrapping a space-like cycle. We show in the next subsection how we may identify the gauge fields and gauge invariant operators of the theory.

3.2 Symmetries

The action of the $\mathrm{U}(1)_m$ factor in the gauge group on the fields (3.3) is generated by β_0^m, inducing the following transformation law for the modes:

$$\begin{aligned} e^{\alpha Q_m} y_n^a &= y_n^a \qquad e^{\alpha Q_m} x_n = x_n + \alpha\nu m x_{n-m}^+ \\ e^{\alpha Q_m} x_n^+ &= x_n^+ \qquad e^{\alpha Q_m} x_n^- = x_n^- + \alpha\nu m x_{n-m} + \frac{1}{2}(\alpha\nu m)^2 x_{n-2m}^+ \,. \end{aligned} \quad (3.7)$$

Although it is a $0+0$ dimensional model, one still considers the above transformations, which are symmetries of the lagrangian, as gauge transformations. Using the σ representation we can combine the transformation into a single gauged U(1) and an associated arbitrary $\Lambda(\sigma)$ living on the S^1 which is the gauge transformation 0-form:

$$\begin{aligned} Y^{a\,\prime} &= Y^a \qquad X^{\prime} = X + \nu\, \partial_\sigma \Lambda\, X^+ \\ X^{+\,\prime} &= X^+ \qquad X^{-\,\prime} = X^- + \nu\, \partial_\sigma \Lambda\, X + \frac{\nu^2}{2} \left(\partial_\sigma \Lambda\right)^2 \, X^+ \,. \end{aligned} \quad (3.8)$$

The gauge transformations above are all connected to the identity of $\mathrm{U}(1)^\infty$. However, there is another element in the group which is disconnected from the identity (we will refer to it as a 'large gauge transformation'). Its action on the fields (3.3) is:

$$U'_{ij} = \sum_{kl} (\beta_1^0)^{-1}_{ik} U_{kl} (\beta_1^0)_{lj}$$

The transformations of this element on σ-represented fields (3.5) are:

$$\begin{aligned} Y^{a\,\prime} &= Y^a \qquad X^{\prime} = X + \nu\, X^+ \\ X^{+\,\prime} &= X^+ \qquad X^{-\,\prime} = X^- + \nu\, X + \frac{\nu^2}{2}\, X^+ \,. \end{aligned} \quad (3.9)$$

These transformations (and all successive transformations generated by it) can be viewed as a modification of (3.8) by allowing a specific non-periodic boundary condition for the 0-form $\Lambda(\sigma)$.

The action (3.6) is also invariant under translations in σ, this is the quantum U(1) symmetry which ensures the conservation of winding number of the string and the momentum in the T dual picture.

3.3 Moduli space

As a check of the formalism we will verify that the classical moduli space becomes the position of a single instanton in the fundamental domain of the null orbifold. The equations of motion derived are:

$$\frac{d}{d\sigma}\left[(X^+)^2\,\frac{dY^a}{d\sigma}\right] = 0 \tag{3.10a}$$

$$\frac{d}{d\sigma}\left[(X^+)^2\,\frac{dX^+}{d\sigma}\right] = 0 \tag{3.10b}$$

$$\left[\frac{d^2(X^+)^2}{d\sigma^2} - \left(\frac{d}{d\sigma}X^+\right)^2\right] X = 0 \tag{3.10c}$$

$$\begin{aligned} X^+ \left[X^+\frac{d^2X^-}{d\sigma^2} - \frac{1}{2}\frac{d^2(X^2)}{d\sigma^2} + \sum_a (\dot{Y}^a)^2\right] + \\ + \frac{d}{d\sigma}\left[X^2\frac{dX^+}{d\sigma} + \frac{\nu^2}{6}\left(\frac{dX^+}{d\sigma}\right)^3\right] = 0\,. \end{aligned} \tag{3.10d}$$

Solving the second and third equations we see that $X^+ = const.$ By gauge invariance we can make X constant (for a nonzero X^+). Then the equations of motion and periodicity constrain for X^- and Y to be constants. The large gauge transformation (3.9) is still not fixed, so the moduli space is:

$$\mathcal{M} = \{X^+, X^-, X, Y^3, \dots Y^9\} \Big/ \left\{\begin{array}{l} X \cong X + \nu X^+ \\ X^- \cong X^- + \nu X + \frac{\nu^2}{2}X^+ \end{array}\right\}. \tag{3.11}$$

The above solution is the Higgs branch and as expected it is the null-orbifold.

The analog of fractional branes (Coulomb branch) is more difficult to understand. We set $X^+ = 0$ and combine the small and large gauge transformations in

$$X^-(\sigma) \cong X^-(\sigma) + \nu\tilde{\Lambda}(\sigma)X(\sigma)\,, \qquad \int_0^{2\pi}\frac{d\sigma}{2\pi}\tilde{\Lambda} \in \mathbb{Z} \tag{3.12}$$

where $\tilde{\Lambda}$ is a periodic function.

For $X(\sigma) \neq 0$ we choose a gauge fixed solution by setting $X^- = constant$ identified by the shifts:

$$X^- = X_0^-\,, \qquad X^- \cong X^- + \nu\left(\frac{1}{2\pi}\int\frac{d\sigma}{X(\sigma)}\right)^{-1}. \tag{3.13}$$

In addition to this X^-, the Coulomb branch is parameterized by $X(\sigma)$ and $Y(\sigma)$ arbitrary functions of σ. Therefore the solutions allow a separation of the fractional branes both in the singular plane and in the transverse directions. Note, however, that the gauge symmetry is completely broken for $X \neq 0$. This is different from the fractional branes of the $\mathbb{R}\times\mathbb{C}/\mathbb{Z}_N$ orbifold, we identify this peculiarity as arising from the existence of a codimension 1 singular plane.

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Pursuing the analogy of T duality, we expect that there is a field redefinition which brings the variables to the form of some gauge fields. Indeed, for $X^+ \neq 0$ we define the fields

$$A \equiv \frac{X}{X^+} \qquad Y^+ = X^+ \qquad Y = X^- - \frac{1}{2}\frac{X}{X^+} \tag{3.14}$$

Y^+,Y are gauge invariant and $A \cong A + \nu\partial_\sigma\Lambda$. Of course, in the action expressed using these variables the gauge field is decoupled from the gauge invariant operators. In addition, it confirms the geometric picture we have. We know of space-like cycle the null orbifold possesses. The gauge field is exactly the coordinate which parameterizes the space-like cycles. On the singular plane, we encounter the same phenomenon, the field

$$A^{\rm sing} = \frac{X^-}{X} \tag{3.15}$$

is again the natural gauge field in agreement with the existence of null cycles.

4. Twisted closed string condensation in the null-orbifold

This section contains the main result of the paper. We show that after condensing a closed twisted sector state, the $D(-1)$ action flows to that of a $\mathbb{Z}_N$ orbifold (at the limit of infinite boost which we elaborate below).

4.1 The action after closed string condensation

We denote by T_k the modulus squared of the k twisted sector closed string state, $k \neq 0$. This choice follows [47]. We repackage the twisted condensate field using the σ variable as

$$T(\sigma) = \sum_{k\neq 0} T_k e^{ik\sigma}\,. \tag{4.1}$$

The quantum U(1) symmetry of the $\mathbb{Z}$ orbifold is given by shifts in σ, and $T(\sigma)$ breaks it with the appropriate charges. In order to flow to a $\mathbb{Z}_N$ orbifold, we turn on only T_k for $k = N\cdot\mathbb{Z}$, i.e, $T(\sigma) = T(\sigma + 2\pi/N)$.

We also define the function $\mathrm{U}(\sigma)$ to be the double integral of T which is periodic and integrates to zero

$$\mathrm{U}(\sigma) = \int\int^{\sigma} T(\sigma) = -\sum_{k\neq 0}\frac{T_k}{k^2}e^{ik\sigma}\,. \tag{4.2}$$

The couplings of the twisted condensate to open string fields are determined uniquely from three properties. The first is locality in the σ variable. The second is that T couples to a quadratic form of the X's. By gauge invariance the possibilities are $X^+(\sigma)X^+(\sigma)$ and $X^2 - 2X^+X^-$. The third requirement is that since the background is invariant under translations in X^-, we can choose the twisted condensate field to have a similar property. This determines the coupling to be

$$\int d\sigma\, T(\sigma)X^+(\sigma)X^+(\sigma)\,. \tag{4.3}$$

This is also the result obtained from taking the limit of the $\mathbb{Z}_N$ orbifold. As in [47], it means that the twisted sector closed string state couples to the open string field which measures the effective distance from the singularity.

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It is convenient to change variables to the following gauge invariant ones by

$$(X^+, X^-, X) \to (X^+, L)\,, \qquad L(\sigma) = 2X^+(\sigma)X^-(\sigma) - X^2(\sigma)\,. \tag{4.4}$$

Using these variables the action becomes

$$\mathrm{S} = \int_0^{2\pi} \left[(X^+)^2 (\dot{Y}^a)^2 - \frac{1}{3}\frac{d(X^+)^3}{d\sigma}\left(\frac{d}{d\sigma}\left(\frac{L}{X^+}\right) + \frac{2}{X^+}\dot{U}(\sigma)\right) - \frac{\nu^2}{12}\left(\dot{X}^+\right)^4 \right]. \tag{4.5}$$

The equations of motions, in terms of the gauge invariant variables are:

$$\frac{d}{d\sigma}\left[(X^+)^2 \frac{dY^a}{d\sigma}\right] = 0 \tag{4.6a}$$

$$\frac{d}{d\sigma}\left(X^+ \frac{dX^+}{d\sigma}\right) + \left(\frac{dX^+}{d\sigma}\right)^2 = 0 \tag{4.6b}$$

$$X^+\left[\sum_a \left(\frac{dY^a}{d\sigma}\right)^2 + \frac{1}{2}\frac{d^2 L}{d\sigma^2} + T(\sigma)\right] - \frac{d}{d\sigma}\left[\left(L\,\frac{dX^+}{d\sigma}\right) - \frac{\nu^2}{6}\left(\frac{dX^+}{d\sigma}\right)^3\right] = 0\,. \tag{4.6c}$$

Using the fact that X^+ and Y^a have to be constant by the first two equations (and periodicity conditions), we can reduce the last equation to:

$$X^+\left[\frac{1}{2}\frac{d^2 L}{d\sigma^2} + T(\sigma)\right] = 0 \tag{4.7}$$

and the general solution is:

$$X^+(\sigma) = X_0^+\,, \qquad L(\sigma) = L_0 - 2\,\mathrm{U}(\sigma) \tag{4.8}$$

In the next subsection we will deal carefully with the IR action around this solution, but we can already see the glimpses of $\mathbb{Z}_N$ orbifold. Given a value of the gauge invariant X_0^+ and L_0 we have an entire gauge orbit of the gauge symmetry (3.8). For generic values of X_0^+ and L_0, the gauge group is completely broken on this orbit. The maximal unbroken gauge group occurs on the orbit

$$X_0^+ = 0\,, \qquad L_0 = \bar{L}_0 \equiv 2 \min_{\sigma\in[0,2\pi]} \mathrm{U}(\sigma) \tag{4.9}$$

where it is $\mathrm{U}(1)^{N-1}$. This point corresponds to bringing as many $D(-1)$ instantons as possible close to the singularity, and the symmetry suggests $D(-1)$ instantons near a $\mathbb{Z}_N$ singularity.

To show this we note first that for $X^+ \neq 0$ the gauge symmetry is completely broken due to the transformation $X \to X + \nu\partial\Lambda X^+$. For $X^+ = 0$, X is gauge invariant and the transformation of X^- (3.12):

$$X^- \to X^- + \nu\tilde{\Lambda}X = X^- + \nu\tilde{\Lambda}\sqrt{-L}\,.$$

Note that $L \leq 0$ everywhere for $X^+ = 0$. We see that if L_0 attains its value from (4.9), there are N points[10] where $L = 0$, and hence $X = 0$, and the symmetry is restored. We identify the gauge symmetry as

$$\Lambda \propto \delta(\sigma - \sigma_i)\,, \qquad \text{for each } \sigma_i \text{ such that } L(\sigma_i) = X(\sigma_i) = 0\,.$$

The constraint $\int \tilde{\Lambda} \in 2\pi\mathbb{Z}$ removes one transformation to obtain the gauge symmetry $\mathrm{U}(1)^{N-1}$ of $D(-1)$ instantons in $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ like orbifold.

Although this is an indication, it can not be taken to be the complete story. This is so because the localized transformations as we have written them are difficult to extend to the entire Higgs branch (where the gauge symmetry is generically broken). In this case we have to smear the localized gauge transformations (to avoid $\delta^2(\sigma - \sigma_0)$ terms), and there seems to be considerable arbitrariness in how to do so.

4.2 The "IR" theory

Since the action is not positive definite, we first need to clarify the notion of "IR" physics. By this we mean that we separate the degrees of freedom in the action into slow and fast variables, where for the latter we carry out a stationary phase approximation. To obtain a clear separation of scales we write the twisted condensate field as

$$T(\sigma) = M \cdot \hat{T}(\sigma)$$

and work in the scaling of $\hat{T}$ fixed and $M \to \infty$.

We are interested in working slightly off-shell — i.e, we do not impose the equations of motion, but restrict our attention to fluctuations which have finite action (does not scale with M). We also would like to work near the singularity. To do so, we need to work close to the special solution (4.9) for which the $\mathrm{U}(1)^{N-1}$ is unbroken. This means that we should take

$$L(\sigma) + \delta L(\sigma) = (\bar{L}_0 - 2\,\mathrm{U}(\sigma)) + \delta L(\sigma)\,. \tag{4.10}$$

For this class L scales linearly in M in all of the interval (due to the term in the parenthesis) except in the N points σ_i, where its value is held fixed as $M \to \infty$ (and determined by δL). Expanding the action to quadratic order in δL, δX^+ and δY around this solution, we obtain

$$\mathrm{S}_{\text{fluc}} = \int \frac{d\sigma}{2\pi} \left[(X_0^+)^2 (\delta \dot{Y}^a)^2 - (X_0^+) \frac{d(\delta X^+)}{d\sigma} \frac{d(\delta L)}{d\sigma} + L \left(\frac{d(\delta X^+)}{d\sigma} \right)^2 \right]. \tag{4.11}$$

In the intervals between the points σ_i, the value of L scales with M, and the variations δX^+ are fast in these intervals. We therefore "integrate out" the variations of X^+ between the σ_i's. These give the constrains that the function $X^+(\sigma)$ is piecewise constant, and may jump at the points σ_i. The function is therefore:

$$X^+(\sigma) = X_0^+ + \delta X_i^+\,, \qquad \sigma_i < \sigma < \sigma_{i+1}\,. \tag{4.12}$$

[10] N is the number of points where $\int\int^{\sigma} T(\sigma')$ reaches it's maximal value.

However (4.11) is not convenient at the points σ_i since the last term is $0 \times \delta^2(\sigma - \sigma_i)$, rather we go back to (4.5). Using the derivative of X^+ we see that the action localizes at the points σ_i.

Next we evaluate the term $\partial_\sigma(L/X^+) - 2\dot{U}/X^+$ at the points σ_i. The first step is to evaluate the behavior of L near the points σ_i. In order to obtain a finite action we need to impose that L/X^+ be continuous at the points σ_i. Otherwise we will have a $\delta^2(\sigma - \sigma_i)$ divergence. We will use the notation

$$L_i^{\pm} = \lim_{\sigma \to \sigma_i \pm} L(\sigma) \tag{4.13}$$

and hence

$$\frac{L_i^+}{X_i^+} = \frac{L_i^-}{X_{i-1}^+}\,. \tag{4.14}$$

The degrees of freedom of L in the intervals between the points σ_i do not appear in the action and can be integrated out. To obtain some physical intuition for the remaining degrees of freedom in L, we impose on L the equations of motion piecewise in these intervals. The solution we obtain in each interval is

$$L(\sigma) = (L_i^+ + 2\,\mathrm{U}(\sigma_i)) + \frac{L_{i+1}^- - L_i^+}{\sigma_{i+1} - \sigma_i}(\sigma - \sigma_i) - 2\,\mathrm{U}(\sigma)\,, \qquad \sigma \in (\sigma_i, \sigma_{i+1})\,. \tag{4.15}$$

Note that the value of U is the same at all the points σ_i.

Applying the expansion (4.15) to the action (4.5), (we rename the fluctuation fields by omitting the δ's in front to unclutter the equations) we find the leading IR action:[11]

$$\mathrm{S}_{IR} = (X_0^+)^2 \int_0^{2\pi} \frac{d\sigma}{2\pi} (\dot{Y}^a)^2 - (X_0^+)^2 \sum_{\sigma_i} \left(X_{i+1}^+ - X_i^+\right) \frac{1}{(\sigma_{i+1} - \sigma_i)} \left(\frac{L_{i+1}^+}{X_{i+1}^+} - \frac{L_i^+}{X_i^+} \right). \tag{4.16}$$

In the next subsection we will match the $X^+ - L$ action to that of the boosted $\mathbb{Z}_N$ singularity.

The situation of the Y's is less clear. To the order that we are working in the open string fields and in the twisted condensate field we do not get a clear separation into N degrees of freedom below a gap. Perhaps this happens at higher orders.

4.2.1 The boosted $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$

We would like to clarify the term "infinitely boosted $\mathbb{Z}_N$" used in the beginning of the section. We start from the action of a fractional D-brane in the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold as

[11] In the calculation of (4.16) we used a "non-symmetrized" value to the integration of a delta function over a discontinuous function

$$\int d\sigma \delta(\sigma - \sigma_i) F(\sigma) \equiv \lim_{\epsilon \to 0-} F(\sigma_i - \epsilon)\,.$$

Any other definition of the integral will result in an equivalent action up to a linear combination of the $L_i^{\pm}$'s.

computed in [47], apply infinite boost similar to the one used in section 2 but keeping the parameter N constant.[12] The action for a $D(-1)$ instanton reads:[13]

$$\begin{aligned}
\mathrm{S}^{\mathbb{Z}_N} &= -\sum_{j=0}^{N-1}\left(X^0_{j+1,j+1}-X^0_{j,j}\right)^2|Z_{j,j+1}|^2+\sum_{a=3}^{9}\sum_{j=0}^{N-1}\left(Y^a_{j+1,j+1}-Y^a_{j,j}\right)^2|Z_{j,j+1}|^2+\\
&\quad+\frac{1}{4}\sum_{j=0}^{N-1}\left(|Z_{j,j+1}|^2-|Z_{j-1,j}|^2\right)^2. \qquad (4.17)
\end{aligned}$$

Expanding the action around a general classical solution in the Higgs branch:

$$X^0_{j,j} = X^0+\delta X^0_{j,j} \qquad (4.18)$$

$$Y^a_{j,j} = Y^a+\delta Y^a_{j,j} \qquad (4.19)$$

$$|Z_{j,j+1}| = |Z|+\delta\,|Z_{j,j+1}| \qquad (4.20)$$

and dropping the δ's we find:

$$\begin{aligned}
\mathrm{S}^{\mathbb{Z}_N}_{\text{fluc}} &= -|Z|^2\sum_{j=0}^{N-1}\left[\left(X^0_{j+1,j+1}-X^0_{j,j}\right)^2-\left(|Z_{j,j+1}|-|Z_{j-1,j}|\right)^2\right]+\\
&\quad+|Z|^2\sum_{a=3}^{9}\sum_{j=0}^{N-1}\left(Y^a_{j+1,j+1}-Y^a_{j,j}\right)^2. \qquad (4.21)
\end{aligned}$$

We boost and rotate the coordinates by:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}X^++X^-\\ X^--X^+\\ \sqrt{2}X\end{pmatrix}=\begin{pmatrix}\frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\,\alpha & 0\\ \sqrt{\beta}\,\alpha & \frac{\alpha}{\sqrt{\beta}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}X^0\\ \mathrm{Re}\,Z\\ \mathrm{Im}\,Z\end{pmatrix},\qquad \alpha=\frac{\sqrt{\beta}}{\sqrt{1-\beta^2}}. \qquad (4.22)$$

Taking the limit[14] $\alpha\to\infty$ we find:

$$\begin{aligned}
X^0 &= \sqrt{2}\alpha X^+ + \frac{X^-}{2\sqrt{2}\alpha}+O(\alpha^{-2})\\
|Z|^2 &= 2\alpha^2(X^+)^2+\left((X)^2-X^+X^-\right)+O(\alpha^{-1}). \qquad (4.23)
\end{aligned}$$

Plugging the transformation into (4.21) and taking the leading order in α (using only the first index of each field):

$$\begin{aligned}
\mathrm{S}^{\text{boosted-}\mathbb{Z}_N}_{\text{fluc}} &= -2\alpha^2(X^+_0)^2\sum_{j=0}^{N-1}\left(X^+_{j+1}-X^+_j\right)\left(\frac{L_{j+1}}{X^+_{j+1}}-\frac{L_j}{X^+_j}\right)+\\
&\quad+2\alpha^2\left|X^+_0\right|^2\sum_{a=3}^{9}\sum_{j=0}^{N-1}\left(Y^a_{j+1,j+1}-Y^a_{j,j}\right)^2+O(\alpha^0). \qquad (4.24)
\end{aligned}$$

The factor α can be absorbed into a rescaling of the coordinates.

[12] In other words we redefine coordinate but do not change the identification of the orbifold.

[13] We are using a slightly different normalization for the Z's than [47].

[14] The explicit parametrization of the transformation was chosen to produce a finite limit.

4.2.2 Relating the IR physics

We found the action of the IR physics around a Higgs branch VEV both in the boosted-$\mathbb{Z}_N$ (4.24) and the null-orbifold (4.16). Focusing on the orbifold directions:

$$\mathrm{S}_{IR}^{\text{boosted-}\mathbb{Z}_N} \supset -(X_0^+)^2 \sum_{j=0}^{N-1} \left(X_{i+1}^+ - X_i^+\right) \left(\frac{L_{i+1}}{X_{i+1}^+} - \frac{L_i}{X_i^+}\right)$$

$$\mathrm{S}_{IR} \supset -(X_0^+)^2 \sum_{\sigma_i} \frac{1}{(\sigma_{i+1} - \sigma_i)} \left(X_{i+1}^+ - X_i^+\right) \left(\frac{L_{i+1}^+}{X_{i+1}^+} - \frac{L_i^+}{X_i^+}\right).$$

The above actions are exactly the same up to rescaling of fields. The action is ill-defined at the singularity, (where the VEV of the X^+ field vanishes) which indicates the emergence of new degrees of freedom, which are the fractional branes (Coulomb branch) together with the expected $\mathrm{U}(1)^{N-1}$ gauge symmetry.

Note, however, that after the rescaling of the fields, the issue of whether one is dealing in a finite boost or an infinite boost is subleading in the boost parameter. In order to see this effect in the flow from the null orbifold to the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ case, we need to look at subleading corrections to the action — in this case subleading in M. We do not know how to do this precisely, and hence can not answer the detailed question of whether the deformed null singularity is the infinitely boosted $\mathbb{R}\times\mathbb{C}/\mathbb{Z}_N$ or boosted by some parameter proportional to a power of M.

Note also that we cannot make a clear study of the IR physics in the Coulomb branch or the Y variables with our probes as already pointed out.

5. Summary and conclusions

The purpose of this paper is to study the geometry of small deformations of the null orbifold, which occur after condensation of twisted sector states. This may be a first step towards understanding both the situation in which twisted sector states are condensed with large VEV's, or the situation of pair creation of twisted sector state, which might be ways in which the singularity might be tamed.

We focused on the case of the null-orbifold since it has a clear relationship to the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold — the former is the large N limit of the latter. We have exhibited, using $D(-1)$ brane probes, some evidence that indeed there is a transition from the null orbifold to the $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ orbifold.

In the case of flows between $\mathbb{R}\times\mathbb{C}/\mathbb{Z}_N$ theories, one can eventually flow to flat space, smoothing out the singularity completely. We expect that a similar situation occurs here — hence we conclude that the null orbifold may be smoothed out by the condensation of twisted sector states.

The intermediate step — of flowing to a $\mathbb{R} \times \mathbb{C}/\mathbb{Z}_N$ — might be interesting by itself. For example, it might be interesting to study the condensation of twisted sector states of the BTZ black hole and examine their relation of the deformed geometry to the microstates of [40].

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A. Calculations of the wave-functions limit

In this appendix we demonstrate (in the untwisted case) the limiting procedure between wave-functions on the $\mathbb{R}\times\mathbb{C}/\mathbb{Z}_N$ orbifold and wave-functions on the null-orbifold. We start with the wave-functions of the $\mathbb{R}\times\mathbb{C}/\mathbb{Z}_N$ orbifold in the static coordinates (2.9)

$$\Psi^N_{k,l} = Ne^{-ik^0x^0+iN\phi l}J_{Nl}(2u)\,, \qquad ue^{i\phi} \equiv -i\bar{k}z\,. \tag{A.1}$$

The limit procedure is defined in (2.12) and (2.13), the boost matrix is

$$M_N = \begin{pmatrix} \frac{a}{\sqrt{a^2-b^2}} & 0 & \frac{b}{\sqrt{a^2-b^2}} \\ \frac{b}{\sqrt{a^2-b^2}} & 0 & \frac{a}{\sqrt{a^2-b^2}} \\ 0 & -1 & 0 \end{pmatrix}, \qquad \text{with } a,b = \frac{\sqrt{1\pm 1/N^2}}{\sqrt{2}}\,. \tag{A.2}$$

We take care of each term of (A.1) separately. First the phase factors:

$$\begin{aligned} e^{-ik^0x^0} &\xrightarrow{\text{boost}} e^{iN^2(ak_0+bk_1)(ax^0-bx^1)} = \\ &= \exp\left[\frac{i}{2}N^2(k_0+k_1)(x^0-x^1)+\frac{i}{2}k_0x^0+\frac{i}{2}k_1x^1+O\Big(\frac{1}{N}\Big)\right] \end{aligned} \tag{A.3}$$

$$\begin{aligned} e^{iN\phi l} &= \left(-\frac{k\bar{z}}{\bar{k}z}\right)^{-\frac{Nl}{2}} \xrightarrow{\text{boost}} \left[-\frac{\left(\frac{i(ak_1+bk_0)}{\sqrt{a^2-b^2}}-k_2\right)\left(\frac{i(bx^0-ax^1)}{\sqrt{a^2-b^2}}-x^2\right)}{c.c}\right]^{-\frac{Nl}{2}} \\ &= (-1)^{\frac{Nl}{2}}\left[\frac{-(k_0+k_1)(x^0-x^1)-i\sqrt{2}\,\frac{x^2(k_0+k_1)+k_2(x^0-x^1)}{N}+O(1/N^2)}{c.c}\right]^{-\frac{Nl}{2}}. \end{aligned} \tag{A.4}$$

The large N limit is easily calculated via $(1+x/N)^N \to e^x$ and reduces to

$$e^{iN\phi l} \xrightarrow{\text{boost}} (-1)^{\frac{Nl}{2}}\exp\left[-il\sqrt{2}\,\frac{x^2(k_0+k_1)+k_2(x^0-x^1)}{(x^0-x^1)(k_1+k_0)}+O\Big(\frac{1}{N}\Big)\right]. \tag{A.5}$$

The remaining part of the story is the Bessel function. We quote the asymptotic expansion of the Bessel function from [54] which shall play a major role

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma(\nu+2k+\frac{1}{2})}{(2k)!\Gamma(\nu-2k+\frac{1}{2})}+R_1\right]- \\ &\quad -\sqrt{\frac{2}{\pi z}}\sin\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma(\nu+2k+\frac{3}{2})}{(2k+1)!\Gamma(\nu-2k-\frac{1}{2})}+R_2\right]. \end{aligned} \tag{A.6}$$

Where for $n > \nu/2 - 1/4$ the remainders satisfy

$$|R_1| < \left|\frac{\Gamma(\nu+2n+1/2)}{(2z)^{2n}(2n)!\Gamma(\nu-2n+1/2)}\right| \tag{A.7}$$

$$|R_2| < \left|\frac{\Gamma(\nu+2n+3/2)}{(2z)^{2n+1}(2n+1)!\Gamma(\nu-2n-1/2)}\right|. \tag{A.8}$$

The Bessel functions in (A.1) transforms under the boost:

$$J_{Nl}\left[\sqrt{((k^1)^2+(k^2)^2)\,((x^1)^2+(x^2)^2)}\right] \xrightarrow{\text{boost}} = \tag{A.9}$$
$$= J_{Nl}\left[N^2\left|x^+k^+\right| + \frac{(k^2x^+)^2 - k^+k^-(x^+)^2 - (k^+)^2x^+x^- + (k^+x^2)^2}{2\left|x^+k^+\right|} + O\left(\frac{1}{N}\right)\right].$$

Using the useful limit

$$\lim_{z\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}z^{-a} = 1\,.$$

We observe that the terms in the Bessel function expansion don't scale as powers of N but only of n and hence no terms can be dropped. Fortunately, we are able to re-sum the series (A.6) in the $N\to\infty$ limit. As emphasized in (A.6) we should carefully estimate the remainder. Let $\nu = Nl$ then we take n such that (at least) $\nu = 2n$ and also $z = \frac{\alpha}{2}n^2$ for some constant α. Estimating the remainder we obtain

$$\left|\frac{\Gamma(\nu+2n+1/2)}{(2z)^{2n}(2n)!\Gamma(\nu-2n+1/2)}\right| = \left|\frac{\Gamma(4n+1/2)}{(\alpha n^2)^{2n}(2n)!\Gamma(1/2)}\right| < \frac{1}{\alpha^{2n}}\frac{(4n)^{2n}}{(n^2)^{2n}} = \left(\frac{4}{\alpha}\right)^{2n}\left(\frac{1}{n}\right)^{2n}.$$

These are preferable circumstances and it holds that

$$\lim_{k\to\infty}\sum_{l=0}^{k}P(l,k) = \sum_{l=0}^{\infty}P(l,\infty)\,.$$

Which means that

$$\lim_{N\to\infty}J_{Nl}(N^2\left|x^+k^+\right|+A) = \lim_{N\to\infty}\sqrt{\frac{2}{\pi N^2\left|x^+k^+\right|}}\cos\left(N^2\left|x^+k^+\right| + A - \frac{\pi}{2}Nl - \frac{\pi}{4}\right)\cdot$$
$$\cdot\left[\sum_{k=0}^{\lceil Nl/2\rceil-1}\frac{(-1)^k}{(2N^2x^+k^+)^{2k}}\frac{\Gamma(Nl+2k+\frac{1}{2})}{(2k)!\Gamma(Nl-2k+\frac{1}{2})} + R_1\right] -$$
$$-2^{nd}\text{term} \tag{A.10}$$

where

$$A \equiv \frac{(k^2x^+)^2 - k^+k^-(x^+)^2 - (k^+)^2x^+x^- + (k^+x^2)^2}{2\left|x^+k^+\right|} + O\left(\frac{1}{N}\right).$$

Taking the limit inside the square brackets gives

$$
\begin{aligned}
&= \left[\lim_{N\to\infty}\sqrt{\frac{2}{\pi N^2\left|x^+k^+\right|}}\cos\left(N^2\left|x^+k^+\right| + A - \frac{\pi}{2}Nl - \frac{\pi}{4}\right)\right]\times \\
&\quad\times\left[\sum_{k=0}^{\infty}\frac{(-1)^k l^{4k}}{(2k)!(2x^+k^+)^{2k}}\right] - 2^{nd}\text{term} \\
&= \left[\lim_{N\to\infty}\sqrt{\frac{2}{\pi N^2 x^+k^+}}\cos(N^2x^+k^+ + A - \frac{\pi}{2}Nl - \frac{\pi}{4})\right]\cos\left(\frac{l^2}{2\left|x^+k^+\right|}\right) - \\
&\quad - \left[\lim_{N\to\infty}\sqrt{\frac{2}{\pi N^2\left|x^+k^+\right|}}\sin\left(N^2\left|x^+k^+\right| + A - \frac{\pi}{2}Nl - \frac{\pi}{4}\right)\right]\sin\left(\frac{l^2}{2\left|x^+k^+\right|}\right) \\
&= \lim_{N\to\infty}\sqrt{\frac{2}{\pi N^2|x^+k^+|}}\cos\left(N^2\left|x^+k^+\right| + A - \frac{\pi}{2}Nl - \frac{\pi}{4} + \frac{l^2}{2\left|x^+k^+\right|}\right).
\end{aligned}
\tag{A.11}
$$

Combining the pieces of the full wave function together

$$
\begin{aligned}
&\Psi^{\text{boosted}}_{k;l}(x^+,x^-,x^2) = \\
&\quad = \sqrt{\frac{2}{\pi\left|x^+k^+\right|}}e^{\pi i\frac{Nl}{2}}e^{-il\frac{x^2k^+-x^+k_2}{x^+k^+}}e^{\left[-iN^2k^+x^+-\frac{i}{2}\left(k^+x^-+k^-x^+\right)+O\left(\frac{1}{N}\right)\right]}\times \\
&\qquad\times\cos\Bigg[N^2\left|x^+k^+\right| - \frac{\pi}{2}Nl - \frac{\pi}{4} + \frac{l^2}{2\left|x^+k^+\right|} + \\
&\qquad\qquad + \frac{(k^2x^+)^2 - k^+k^-(x^+)^2 - (k^+)^2x^+x^- + (k^+x^2)^2}{2\left|x^+k^+\right|} + \\
&\qquad\qquad + O\left(\frac{1}{N}\right)\Bigg] \\
&\quad = \frac{(\text{sign}(k^+x^+))^{Nl}}{\sqrt{2\pi i x^+k^+}}\Bigg[e^{\left[i\frac{\left(l-x^2k^++x^+k^2\right)^2}{2x^+k^+}+ik^2x^2-ik^+x^--ik^-x^++O(N^{-1})\right]} + \\
&\qquad\qquad + (-)^{Nl+1}e^{\left[-i\frac{(l+x^2k^+-x^+k^2)^2}{2x^+k^+}-ik^2x^2-i2N^2k^+x^++O(N^{-1})\right]}\Bigg].
\end{aligned}
\tag{A.12}
$$

This is the expression (up to trivial algebra) quoted in the text (2.14). The first exponential is the wave function of the null-orbifold and the second produces an expression which is interpreted as a vanishing distribution. In order to prove the last statement, we study the integration of the second exponential in the boosted wave-function with a test function $g(x^+,x)$ in the large-N limit:

$$
\int dx^+dx^2 g(x^+,x^2)\frac{1}{\sqrt{2\pi i x^+k^+}}\exp\left[-i\frac{\left(l+x^2k^+\right)^2}{2x^+k^+} - i2N^2k^+x^+\right].
$$

Without loss of generality we substituted $k^2=0$. The x^+ integration can now be evaluated using a saddle point method:

$$\begin{aligned}
&\int dx^+ dx^2 g(x^+, x^2) \frac{1}{\sqrt{2\pi i x^+ k^+}} \times \\
&\times \exp\left[-i\frac{\left(l + x^2 k^+\right)^2}{2x^+ k^+} - i2N^2 k^+ x^+\right] \propto \sum_{\pm} \int dx^2 \frac{1}{Nk^+} g\left(\pm\frac{(l+x^2k^+)}{2Nk^+}, x^2\right) e^{\mp 2iN(l+x^2k^+)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \propto \sum_{\pm} \frac{\widetilde{G}^N_{\pm}(\pm 2Nk^+)}{N} \xrightarrow{N\to\infty} 0\,. \qquad \text{(A.13)}
\end{aligned}$$

Where $\widetilde{G}^N_{\pm}$ is the Fourier transform (with respect to x^2) of:

$$G^N_{\pm} \equiv g\left(\pm\frac{(l+x^2k^+)}{2Nk^+}, x^2\right).$$

For a large class of functions[15] $g(x^+, x^2)$ (which are in particular $\mathbb{L}_2$) the limit (A.13) is indeed zero which proves our claim.

B. First quantization of the superstring on $\mathbb{R}^{1,2}/\Gamma$

The orbifolds in mind are defined by a flat space CFT with a quotient by a twist:

$$\begin{aligned}
ds^2 &= -d(x^0)^2 + d(x^1)^2 + d(x^2)^2 + dx_{\parallel}^2 \\
X &\cong e^{2\pi \hat{J}} X = \left[e^{2\pi \mathcal{J}}\right]^{\mu}{}_{\nu} X^{\nu}\,, \qquad \mathcal{J} \in \mathrm{SO}(1,2)\,. \qquad \text{(B.1)}
\end{aligned}$$

We discuss 3 cases according the 3 classes of SO(1,2):

- The Milne orbifold (J is hyperbolic): $J_{\Delta} = 2\pi\Delta\hat{J}_{02}$.
- The Null-orbifold (J is parabolic): $J_v = \frac{2\pi v}{\sqrt{2}}\left(J_{02} + J_{12}\right)$
- The $\mathbb{Z}_N$ orbifold (J is elliptic): $J_N = \frac{2\pi}{N} J_{12}$.

The operators above can be defined using the matrices in (x^0, x^1, x^2) basis:

$$\mathcal{J}_{\Delta} = \Delta\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathcal{J}_v = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathcal{J}_N = \frac{1}{N}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The worldsheet action is:

$$S = \frac{1}{4\pi}\int d^2z\, \frac{2}{\alpha'}\eta_{\mu\nu}\left(\partial X^{\mu}\bar{\partial}X^{\nu} + \eta_{\mu\nu}\psi^{\mu}\bar{\partial}\psi^{\nu} + \eta_{\mu\nu}\tilde{\psi}^{\mu}\partial\tilde{\psi}^{\nu}\right). \qquad \text{(B.2)}$$

[15]We didn't carry a full classification of the functions $g(x^+, x^2)$ that have the above property. A large enough set of examples are polynomials in x^+, x^2 multiplied by decaying exponentials and gaussians.

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And the monodromies:[16]

$$\begin{aligned} X(\sigma+2\pi,\tau) &= e^{2\pi w\mathcal{J}}X(\sigma,\tau) \\ \psi\left(ze^{2\pi i}\right) &= e^{-2\pi i\left((\nu-\frac{1}{2})\mathbb{1}+iw\mathcal{J}\right)}\psi(z) \\ \tilde{\psi}\left(\bar{z}e^{-2\pi i}\right) &= e^{2\pi i\left((\tilde{\nu}-\frac{1}{2})\mathbb{1}-iw\mathcal{J}\right)}\tilde{\psi}(\bar{z}) \end{aligned} \tag{B.3}$$

where ν and $\tilde{\nu}$ take the values of 0 for R-sector and $\frac{1}{2}$ for NS-sector and $w\in\mathbb{Z}$ is the twisted sector number.

The bosonic part mode expansion in analogy to [26] ($w\neq 0$):

$$\begin{aligned} \sqrt{\frac{2}{\alpha'}}X^\mu(\sigma,\tau) &= \sqrt{\frac{2}{\alpha'}}\left[e^{w\mathcal{J}\sigma}\right]^\mu_{\ \rho}X_z^\rho(\alpha_0,\tilde{\alpha}_0;\tau)+i\sum_{n\neq 0}\left[\frac{e^{-i(n+iw\mathcal{J})(\sigma+\tau)}}{n+iw\mathcal{J}}\right]^\mu_{\ \rho}\alpha_n^\rho + \\ &\quad +i\sum_{n\neq 0}\left[\frac{e^{i(n-iw\mathcal{J})(\sigma-\tau)}}{n-iw\mathcal{J}}\right]^\mu_{\ \rho}\tilde{\alpha}_n^\rho\,. \end{aligned} \tag{B.4}$$

The zero-mode part is

$$\text{Null}\,,\quad X_z{}^\mu(\tau) = \cosh\!\big(w\tau\mathcal{J}\big)^\mu{}_\nu x^\mu + \left[\mathbb{1}+\frac{1}{2}\cosh\!\big(w\tau\mathcal{J}\big)\right]^\mu_{\ \nu}\frac{2\alpha'\tau}{3}p^\nu \tag{B.5a}$$

$$\text{Other}\,,\quad X_z{}^\mu(\tau) = \cosh\!\big(w\tau\mathcal{J}\big)^\mu{}_\nu x^\mu + \left[(w\mathcal{J})^{-1}\sinh(w\mathcal{J}\tau)\right]^\mu_\nu\,\alpha' p^\nu\,. \tag{B.5b}$$

Using the euclidean world-sheet, the left/right moving parts can be expanded:

$$\partial X(z) = -iz^{\,-(1+iw\mathcal{J})}(w\mathcal{J}x+\alpha' p)-i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}z^{\,-(n+1+iw\mathcal{J})}\alpha_n \tag{B.6a}$$

$$\bar{\partial}X(\bar{z}) = i\bar{z}^{\,-(1+iw\mathcal{J})}(w\mathcal{J}x-\alpha' p)+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\bar{z}^{\,-(n+1-iw\mathcal{J})}\tilde{\alpha}_n \tag{B.6b}$$

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu}\left[\frac{1}{z^{r+\frac{1}{2}+iw\mathcal{J}}}\right]^\mu_{\ \rho}\psi_r^\rho \tag{B.6c}$$

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{s\in\mathbb{Z}+\tilde{\nu}}\left[\frac{1}{\bar{z}^{s+\frac{1}{2}-iw\mathcal{J}}}\right]^\mu_{\ \rho}\tilde{\psi}_s^\rho\,. \tag{B.6d}$$

The commutation relations are calculated form the above expansions using the OPE and canonical quantization relations (needed for the quasi-zero modes).

$$\begin{aligned} [\tilde{\alpha}_n^\mu,\tilde{\alpha}_m^\nu] &= \delta_{n+m}(n\eta-iw\mathcal{J}\eta)^{\mu\nu} \qquad \{\psi_r^\mu,\psi_{r'}^\nu\} = \delta_{r+r',0}\eta^{\mu\nu} \\ [\alpha_n^\mu,\alpha_m^\nu] &= \delta_{n+m}(n\eta+iw\mathcal{J}\eta)^{\mu\nu} \qquad \left\{\tilde{\psi}_s^\mu,\tilde{\psi}_{s'}^\nu\right\} = \delta_{s+s',0}\eta^{\mu\nu}\,. \end{aligned} \tag{B.7}$$

Where we adopted the following definition of quasi-zero modes:[17]

$$\alpha_0 = \frac{\alpha' p+w\mathcal{J}x}{\sqrt{2\alpha'}} \qquad \tilde{\alpha}_0 = \frac{\alpha' p-w\mathcal{J}x}{\sqrt{2\alpha'}} \qquad [\alpha_0,\tilde{\alpha}_0]=0\,. \tag{B.8}$$

[16] Remember $z=e^{i(\sigma+\tau)}$, $\bar{z}=e^{-i(\sigma-\tau)}$.

[17] Note that the above definition do not consist of a full set of zero-mode operators. This is similar to flat space where $\alpha_0^\mu=\tilde{\alpha}_0^\mu=\sqrt{\frac{\alpha'}{2}}p^\mu$.

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The Virasoro generators (matter part) are

$$L_n^\psi = \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}\left[(2r-n)\eta + iw\left(\eta\mathcal{J}-\mathcal{J}^T\eta\right)\right]_{\mu\nu} {}^\circ_\circ\psi^\mu_{n-r}\psi^\nu_r{}^\circ_\circ + a^\psi(w,\nu)\delta_{n,0} \tag{B.9a}$$

$$L_n^x = \frac{1}{2}\sum_m {}^\circ_\circ\alpha_m\eta\alpha_{n-m}{}^\circ_\circ + \delta_{n,0}a^x(w)\,. \tag{B.9b}$$

Using $[L_1, L_{-1}] = 2L_0$ we find:

$$a^x = \frac{w^2}{4}\,\mathrm{Tr}(\mathcal{J}^2) + \left\langle \mathrm{vac}\middle|\frac{1}{2}\eta_{\mu\rho}\alpha_0^\mu\alpha_0^\rho\middle|\mathrm{vac}\right\rangle_w \tag{B.10a}$$

$$a^\psi_{NS} = -\frac{w^2}{4}\,\mathrm{Tr}(\mathcal{J}^2) \tag{B.10b}$$

$$a^\psi_R = \frac{D}{16} - \frac{w^2}{4}\,\mathrm{Tr}(\mathcal{J}^2) + \left\langle \mathrm{vac}\middle|\left(\frac{iw}{2}\eta\mathcal{J}\right)_{\mu\sigma}\psi_0^\mu\psi_0^\sigma\middle|\mathrm{vac}\right\rangle_{w,R}. \tag{B.10c}$$

In order not to break worldsheet supersymmetry the quantization scheme for the bosonic zero modes and the R-sector fermionic zero modes must obey

$$a^\psi_R + a^x = \frac{D}{16}\,.$$

Applying this constraint we can fix the quantization scheme for the different classes of orbifolds (using the already discussed scheme of the bosonic part). The zero-point energies can then be calculated according to:

- **Elliptic orbifolds class:** find d+1 vectors that diagonalize the matrix $iw\mathcal{J}\eta$. Vectors corresponding to positive eigenvalue λ annihilate the vacuum. Vectors corresponding to zero eigenvalue are 'standard' zero-modes which annihilate the vacuum (these are momentum operators). The bosonic zero point energy have a contribution from negative eigenvalue modes, $\frac{1}{2}\lambda_i$ for each negative eigenvalue (the R-sector will have the opposite contribution).

- **Hyperbolic orbifolds class:** find d+1 vectors that diagonalize the matrix $iw\mathcal{J}\eta$, their eigenvalues are pure imaginary. Vectors corresponding to zero eigenvalue are 'standard' zero-modes which annihilate the vacuum (actually these are momentum operators). Vector corresponding to (non-vanishing) pure imaginary zero mode should be treated as in [26] to eliminate imaginary contribution to the zero point energies.

- **Parabolic orbifolds class:** either by taking the limit from the hyperbolic or elliptic cases we set the zero-point energies to zero (the orbifold part).

The different zero-point energies are calculated for the representatives of the classes ($\mathbb{Z}_N$, Null and Milne orbifolds):

	a^X	a^ψ_{NS}	a^ψ_R	a^g_{NS}	a^g_R
Flat	0	0	$D/16$	$-1/2$	$-5/8$
$\mathbb{Z}_n$	$w/2N(1-w/N)$	$w^2/2N^2$	$D/16 - w/2N(1-w/N)$	$-1/2$	$-5/8$
Milne	$\frac{w^2\Delta^2}{2}$	$-\frac{w^2\Delta^2}{2}$	$D/16-\frac{w^2\Delta^2}{2}$	$-1/2$	$-5/8$
Null	0	0	$D/16$	$-1/2$	$-5/8$

B.1 The vacuum structure

By a simple manipulation of commutation relations we can rewrite L_0 as:

$$L_0 = L_0^{(\mathrm{diag})} + \frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\Sigma^{\nu\mu} - A\,. \tag{B.11}$$

With the Lorentz generators defined as

$$\Sigma^{\mu\nu} \equiv -\frac{i}{2}\sum_{r\in\mathbb{Z}+\nu}\left[\psi^\mu_r, \psi^\nu_{-r}\right], \qquad \frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\left[\Sigma^{\nu\mu}, \psi^\sigma_r\right] = -iw\mathcal{J}^\sigma{}_\mu\psi^\mu_r\,.$$

The diagonal part of L_0:

$$L_0^{(\mathrm{diag})} = \frac{1}{2}\sum_m {}^\circ_\circ\alpha^\mu_{-m}\alpha_{m\mu}{}^\circ_\circ + \frac{1}{2}\sum_{r\in\mathbb{Z}+\nu} r\,{}^\circ_\circ\psi^\mu_{-r}\psi_{r\mu}{}^\circ_\circ + L_0^{\mathrm{ghost}} + a(w,\nu)\,.$$

And the constant A defined as:

$$A(w,\nu) = \left\langle \mathrm{vac}\middle|\left(\frac{iw}{2}\eta\mathcal{J}\right)_{\mu\sigma}\psi^\mu_{-\nu}\psi^\sigma_\nu\middle|\mathrm{vac}\right\rangle_{w,\nu}.$$

The constant A rises from "undoing" the normal ordering of the ψ's. It takes a non vanishing value (equal to $\frac{w}{2N}$) only in the R-sector of the elliptic orbifold where there exist fermion zero modes[18] (and not only quasi-zero modes).

Remembering that a physical state in the CFT must be a zero eigenstate of L_0 we study the effect of the operator $\frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\Sigma^{\mu\nu}$ on the spectrum.

- **Elliptic orbifolds class:** the operator $\frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\Sigma^{\mu\nu}$ is a generator of a rotations group SO(2) in the direction of the orbifold and has real eigenvalues. By changing the charges of the left moving and right moving sides it is possible to generate physical states obeying $L_0 = \widetilde{L}_0 = 0$ of "mixed" type $(\mathrm{NS}, \mathrm{R})$.

  As is well known from the literature [47, 48], in these theories if N is an even integer, untwisted fermions cannot be introduced. By taking N odd one is able to construct type-II theory where the untwisted tachyon is projected out. The physical condition $L_0 = \widetilde{L}_0 = 0$ forces us to consider (NS-, R) sectors for odd ω and there are tachyons in all twisted sectors.

- **Hyperbolic orbifolds class:** the operator $\frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\Sigma^{\mu\nu}$ is a generator of a boost group $SO(1,1)$ in the direction of the orbifold it has pure imaginary eigenvalues. These eigenvalues can be compensated by a suitable bosonic wave-function.

- **Parabolic orbifolds class:** the operator $\frac{w}{2}(\eta\mathcal{J})_{\mu\nu}\Sigma^{\mu\nu}$ is a generator of a null-boost group in the direction of the orbifold. The operator is nilpotent such that L_0 can be written in a Jordan form. Thus the CFT is logarithmic, at this point we cannot determine whether it is possible to find a consistent set of constraints on the spectrum (i.e BRST + GSO + orbifold projection) such that the resulting theory will have no branch cuts (mutual locality between operators) and will be modular invariant.

[18] In the Elliptic case in the sector $w = N/2$ there are fermion zero modes in the NS sector, we ignore this subtlety which is not important for our discussion.

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